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Aligned-field hydromagnetic flow past a slender body

By S. LEIBOVICH
University College London
AND G. S. S. LUDFORD
Cornell University

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This is a continuation of our previous work (1965) on the Sears–Resler–Stewartson controversy, in the context of axially symmetric flow. A new approach is presented using boundary-layer arguments, which remove much of the old complexity.

For resistive bodies of shape $R(x)$ we uncover a similarity with plane airfoils of shape $F(x) = R^2(x)$. As before, Stewartson's slug flows develop fore and aft. For bodies of very high conductivity the Sears–Resler (steady-state) solution turns out to be one possibility. It pertains to bodies (of much higher conductivity than the liquid) into which the initial magnetic field has not diffused.

1. Introduction

Steady flows of perfectly conducting inviscid liquids past thin airfoils were first studied by Sears & Resler (1959). Further consideration by Stewartson (1960) suggested that the sub-Alfvénic steady problem is not uniquely posed. Leibovich & Ludford (1965) re-examined the problem for bodies of finite (or zero) conductivity by first constructing a complete transient solution, and then passing to the steady limit.

This work is a sequel to the latter paper, which will be denoted by L & L, and to which we shall frequently refer.

We treat first the axisymmetric version of the plane problem considered in L & L. Although few modifications are required in the analysis, a different (boundary-layer) approach to the wave regions is presented in § 2. It is more systematic and may allay misgivings about the *ad hoc* character of our previous treatment. No longer do we need to distinguish between the potential and wave parts in these regions (§ 6). § 7 contains a simple derivation of the ultimate motion.

The boundary problem developed in §§ 3 and 5 discloses the similarity between a slender axisymmetric body of shape $R(x)$—see equation (2)—and a plane symmetric airfoil with shape

$$F(x) = R^2(x).$$

This is the main result for resistive bodies.

Next, in §§ 8 and 9, we take up a perfect conductor, since this sheds more light on the Sears–Resler–Stewartson controversy (Stewartson 1960). Two new cases arise corresponding to two possible types of magnetic field before the impulsive start (which leaves them unchanged).

Consider a body of finite conductivity σ_b at rest in a liquid of finite but large conductivity σ_l. When a uniform magnetic field is switched on, it will take times proportional to σ_b and σ_l to become established in the body and liquid respectively. Three situations may be distinguished.

Case 1. σ_b is small compared to σ_l and we wait for the field to become uniform everywhere. Our treatment (§§ 1–7) with $\sigma_l = \infty$ shows that in the subsequent motion two Alfvén waves, one upstream and the other downstream, move out to stop the flow relative to the body in two straight tubes bounded by vortex-current sheets. These are joined by a current-vortex sheet lying on a section of the body aft of its mid-section. The approximation holds if the diffusion time based on σ_l is large compared with the time in which the upstream wave moves a body length.

Case 2. σ_b is at least comparable to σ_l and again we wait for the field to become uniform everywhere. § 8, based on $\sigma_b = \sigma_l = \infty$, shows that the flow is similar to that in case 1 except that both tubes are bigger and their join lies in the liquid, being attached to the body only at the mid-section itself. The approximation holds under the same conditions.

Case 3. σ_b is large compared to σ_l and we wait only for the field to diffuse into the liquid. § 9, based on $\sigma_b = \sigma_l = \infty$, shows that Sears–Resler (1959) potential flow† is established instantaneously. This solution was proposed for insulators, but is in fact valid for extremely good conductors.

In all cases, the applied magnetic field which is established is uniform at infinity and parallel to the body symmetry axis, and to the free stream.

We may expect the flows in cases 2 and 3 to be transitory only. In a time comparable to the σ_b-diffusion time a blurred version of that in 1 will form and persist.

Similar remarks (and the corresponding analysis) could equally well have been developed for the plane airfoil.

2. Formulation

Let Ox be the symmetry axis of the body and y the distance from it. Then the equations appropriate for axisymmetric flow are (cf. L & L (1))

$$u_t + uu_x + vu_y + P_x = \beta^2(hh_x + kh_y), \tag{1a}$$

$$v_t + uv_x + vv_y + P_y = \beta^2(hk_x + kk_y), \tag{1b}$$

$$h_t + uh_x + vh_y = hu_x + ku_y, \tag{1c}$$

$$k_t + uk_x + vk_y = hv_x + kv_y, \tag{1d}$$

$$(yu)_x + (yv)_y = 0, \tag{1e}$$

$$(yh)_x + (yk)_y = 0. \tag{1f}$$

All quantities in (1) are dimensionless with U_0, the free stream, as unit of velocity (u, v); and H_0, applied field, as unit of magnetic intensity (h, k). P is the total pressure, fluid plus magnetic, and is referred to ρU_0^2, where ρ is the density. $\beta^2 = (\mu H_0^2)/(\rho U_0^2)$ is the squared ratio of the Alfvén speed to the fluid speed in

† Yih (1965) has described another way in which this flow can be set up.

undisturbed regions. The permeability, μ, of the fluid is assumed to be the same in the body. c, the body length, is the unit of distance, and c/U_0 that of time.

Locate the origin O at the maximum section. Then, if

$$y = \tau R(x), \quad \text{where} \quad R(0) = 1 \quad (\tau \ll 1) \tag{2}$$

denotes the shape and $x = a-1$, a are the nose and tail, respectively, we have

$$R'(0) = R(a-1) = R(a) = 0.$$

The body is pointed at either end and, for simplicity, convex.

The impulsive start at time $t = 0$ leaves the magnetic field (whether that of case 1–3) unaltered. The fluid is set into potential motion past the body, which is an $O(\tau)$ disturbance of uniform flow. Any vorticity or current which may subsequently appear in the fluid must originate at the body. One expects these quantities to be either convected with the fluid or propagated with the Alfvén-wave velocity. Because the body is slender and the magnetic field essentially uniform, regions of non-zero vorticity and current, if they exist, are anticipated only in the immediate vicinity of the symmetry axis.

The boundary conditions (developed in the next section) show, as in L & L, that an $O(1)$ longitudinal disturbance is required subsequently. Continuity then requires that $\partial/\partial y$: $\partial/\partial x = O(\tau^{-1})$. Intense current and vorticity are propagated away from the body at the Alfvén velocity relative to the flow, forming tubular wave regions in front of and behind the body (for $\beta > 1$).

In these regions we therefore write

$$v = \tau V, \quad k = \tau K, \quad y = \tau Y$$

and assume that all new variables and their derivatives are of order unity. The equations of motion become

$$U_t + UU_x + VU_Y + P_x = \beta^2(HH_x + KH_Y), \tag{3a}$$

$$H_t + UH_x + VH_Y = HU_x + KU_Y, \tag{3b}$$

$$(YU)_x + (YV)_Y = 0, \tag{3c}$$

$$(YH)_x + (YK)_Y = 0, \tag{3d}$$

$$P_Y = \tau^2[\beta^2(HK_x + KK_Y) - (V_t + UV_x + VV_Y)], \tag{3e}$$

where we have written

$$u = U(x, Y, t), \quad h = H(x, Y, t)$$

to distinguish the solution inside the regions from that outside. The Y-induction equation has been omitted since it can be derived from the others.

Equation (3e) implies that

$$P = P(x, t), \tag{4}$$

correct to $O(\tau^2)$. Since the total pressure is continuous everywhere in the fluid and its disturbance is $O(\tau)$ outside the wave regions, P_x is therefore $O(\tau)$, and may be neglected in (3a). It and equation (3b) will be replaced by

$$(U \pm \beta H)_t + (U \mp \beta H)(U \pm \beta H)_x + (V \mp \beta K)(U \pm \beta H)_Y = 0, \tag{5$\mp$}$$

which show that $U \pm \beta H$ is conserved along the lines

$$l_{\mp}: dt = dx/(U \mp \beta H) = dY/(V \mp \beta K) \qquad (6\mp)$$

in (x, Y, t)-space. These are the Alfvén waves mentioned earlier.

The equations governing the wave regions are ($3c, d$), (4), and ($5\mp$).

3. Boundary conditions

At the surface of the body

$$v = \tau R'(x)\, u, \qquad (7)$$

and the normal component of magnetic field is continuous. If the body has finite conductivity the tangential component is also continuous (when the fluid is inviscid and perfectly conducting in the sense of L & L, i.e. vanishing viscosity, vanishing magnetic diffusivity, with the ratio of the first to the second also zero). On the other hand, if the conductivity of the body is so high that it may be assumed infinite, the relevant condition is the vanishing of the electric field. We must now translate these into boundary conditions on the velocity and magnetic field in the fluid.

Finite conductivity

Equation ($1f$) holds everywhere and may be satisfied by setting

$$yh = \psi_y, \quad yk = -\psi_x.$$

The magnetic flux ψ is even in y and so has the form

$$\psi(x,y,t) = \tfrac{1}{2}y^2h(x,y,t) + O(y^4) \qquad (8)$$

in the body (cf. L & L (8)). Outside it is convected with the fluid, and this yields

$$\psi_t + q\psi_s = 0 \qquad (9)$$

on the body. Here q is surface speed and s is arc length along the body surface.

Now, both ψ and h are continuous across the interface. So we may insert (8) into (9) to obtain the equation

$$(R^2h)_t + q(R^2h)_s = O(\tau^2) \qquad (10)$$

for the values of h in the fluid at the body surface. The component k is given in terms of h by

$$k = -\tfrac{1}{2}\tau R h_s + O(\tau^3), \qquad (11)$$

as is easily seen from equation (8). Clearly q can be replaced by u and s by x (now a parameter on the surface) without increasing the error in equations (10) and (11).

In terms of the variables introduced in §2 the conditions (7), (10), and (11) become

$$V = R'U, \quad (R^2H)_t + U(R^2H)_x = 0, \quad K = -\tfrac{1}{2}RH_x. \qquad (12a, b, c)$$

The second of these requires $H_t = -2R'/R$ initially, whereas the initial potential flow gives $H_t = O(\tau)$ (see equation ($3b$)). It is this incompatibility which is resolved by the emission of Alfvén waves.

This all pertains to case 1.

Infinite conductivity

The electric field in the fluid is $vh-uk$ (in suitable units) so that

$$qh_n = 0 \tag{13}$$

at the surface, where h_n is the normal component of magnetic field. Inside the body the magnetic field is frozen.

In case 2 the initial field which persists inside the body is uniform, so that $q = 0$, i.e. $u = v = 0$ at the surface. Continuity of h_n also adds

$$k = \tau R'(h-1).$$

In terms of the new variables the conditions become

$$U = V = 0, \quad K = R'(H-1). \tag{14,a,b,c}$$

In case 3 the initial field in the body is zero, so that $h_n = 0$, and condition (13) is automatically satisfied.

4. A relation between U and H

In this section, which applies equally to cases 1–3, we deduce some of the properties of the lines $l_\mp$ described by (9).

The lines have a direction, that of t increasing. Since they only have meaning in the fluid, they may begin or end only on the body surface (for $t > 0$) or on the plane $t = 0$. It should be emphasized that they apply in the external region of potential flow as well as in the channel. A line which originates at $t = 0$ carries undisturbed values of $U-\beta H$ or $U+\beta H$, but a line originating at the body surface carries whatever values of the pertinent combination of U and H it may have had at the body.

The vectors $(U, V) \pm \beta(H, K)$ point to opposite sides of a stream-line if the normal component, H_n of magnetic field is non-zero. Therefore, at a solid boundary penetrated by the magnetic field, one and only one of the pair $l_\mp$ points into the fluid. A 'wave' consists of a packet of those lines $l_\mp$ carrying disturbances to U and H, i.e. those $l_\mp$-lines originating at the solid boundary. Consequently, one wave consisting of l_+-lines and another of l_--lines cannot intersect on the body surface. It is conceivable that two such waves could intersect in the fluid but, as in two dimensions, the assumption that they do not leads to a consistent solution.

In view of these remarks, we see that U and H are simply related in a wave region. Consider a wave consisting of l_+-lines; since $U+\beta H$ is conserved along l_--lines, we must have

$$U+\beta H = 1+\beta \tag{15+}$$

in the (downstream) wave, because all of the l_--lines crossing the wave originate at $t = 0$, where $U = H = 1$. Similarly,

$$U-\beta H = 1-\beta \tag{15-}$$

in a wave (upstream) consisting of l_--lines. These imply [cf. equations (6 ∓)] that the x-components of the wave velocities are constant; and that they have appropriate signs for $\beta > 1$.

At any given instant the surface of the body is divided into two parts at the transition station. One part is transmitting upstream (l_--lines) and receives the information (15 −) from undisturbed conditions; the other part transmits downstream (l_+-lines) and receives (15 +). The two parts are characterized by the sign of H_n.

5. Solution on the boundary in case 1

The values of U and H on the boundary are determined by equations (12b) and (15 ∓) without further reference to the disturbance in the fluid. Since this problem is identical with that for a plane airfoil with shape function $F(x) = R^2(x)$, we can read off its solution from L & L.

The transition station (which starts at $x = 0$) moves downstream and ultimately comes to rest at $x = x_t$ where

$$R(x_t) = \sqrt{(\beta-1)/(\beta+1)}.$$

The surface is then divided into three parts.

(i) $a-1 < x < 0$. Here $U \to 0$ and $H \to 1-1/\beta$. Transmission was always upstream.

(ii) $0 < x < x_t$. Here $U \to (\beta-1)[(1/R^2(x))-1]$ and $H \to (\beta-1)/\beta R^2(x)$. U increases steadily from 0 to 2, and H from $1-1/\beta$ to $1+1/\beta$. Magnetic lines follow the body contour. Transmission was first downstream but eventually became upstream.

(iii) $x_t < x < a$. Here $U \to 0$ and $H \to 1+1/\beta$. Transmission was always downstream.

In (i) and (iii) the approach is exponential in time at each station; in (ii) it is algebraic.

6. The wave regions. External potential fields

Consider the forward wave. The values of U and H on the boundary are propagated into the fluid according to (6 −), where $U-\beta H = 1-\beta$. To determine U and H in the wave we must therefore know $V-\beta K$, and it is remarkable that this combination can be found without knowing V and K separately. In fact their determination depends on finding U and H first.

Equations (3c), (3d), and (15 −) show that

$$[Y(V-\beta K)]_Y = 0,$$

so that

$$Y(V-\beta K) = R(V_b-\beta K_b). \qquad (16)$$

Here the subscript b denotes values on the body or on the axis which are given in § 3, for any of the cases 1–3. For convenience we define $R \equiv 0$, for x not in $(a-1, a)$.

$V-\beta K$ is therefore a known function of x and t divided by Y. A signal is displaced laterally while abreast of the body but travels horizontally once clear of the nose. The assumption of a thin wave region is therefore not violated if $\tau/(\beta-1) \ll 1$.

We shall not write down the formulae for U and H as functions of x, Y, t. From them V and K are separately determined by the continuity equations $(3c, d)$.

$$YV = RV_b - \int_R^Y \bar{Y} U_x(x, \bar{Y}, t)\, d\bar{Y}, \tag{17a}$$

$$YK = RK_b - \int_R^Y \bar{Y} H_x(x, \bar{Y}, t)\, d\bar{Y}. \tag{17b}$$

For the backward wave change the sign of β.

The edge of either wave region is marked by signals which left the body at $t = 0$ and those emitted by the transition station. Outside there is a potential disturbance of $O(\tau)$ of the free stream and magnetic field, so that

$$u = 1 + \tau(\phi_t + \phi_x)_x, \quad h = 1 + \tau\phi_{xx},$$
$$v = \tau(\phi_t + \phi_x)_y, \quad k = \tau\phi_{xy},$$
$$P = -\tau[\phi_{tt} + 2\phi_{xt} + (1-\beta^2)\phi_{xx}], \quad y\phi_{xx} + (y\phi_y)_y = 0,$$

(cf. Stewartson 1960). Across an edge there is† continuity of (*a*) mass flux, (*b*) magnetic flux, (*c*) total pressure, and (*d*) tangential momentum flux.

Conditions (*a*) and (*b*) amount to continuity of normal velocity and magnetic intensity, respectively, and together fix ϕ.

$$(\phi_t + \phi_x)_y = V + (1-U)\frac{\partial Y_w}{\partial x}, \tag{18a}$$

$$\phi_{xy} = K + (1-H)\frac{\partial Y_w}{\partial x} \tag{18b}$$

on the wave edge.‡ Each of these presents a standard problem in hydrodynamic slender-body theory, which will not be dealt with further here. It is necessary to check that the $\phi_t + \phi_x$ obtained from the first is compatible with the ϕ_x obtained from the second. The details are similar to, and just as mysterious as, those for plane flow given in L & L, so we shall omit them.

Once ϕ is determined and the corresponding P calculated, condition (*c*) fixes the $P(x, t)$ of equation (4). Condition (*d*) gives the perturbed values of $U \pm \beta H$ entering the wave regions, so that it falls on the longitudinal disturbance of $O(\tau)$, so far not discussed. The calculation of this disturbance is lengthy and does not add materially to the present picture. Leibovich's thesis (1965) gives the details for plane flow, which is very similar.

7. Case 1. The ultimate motion. Drag

Consider again the forward wave and now its ultimate shape. The point (x, Y), which at time t receives a signal originating at the body at x_p and time t_p is, according to equation $(6-)$, given by

$$x = x_p - (\beta - 1)(t - t_p), \tag{19a}$$

$$Y^2 = R^2(x_p) - \frac{1}{\beta - 1}\int_{x_p}^{x} \theta[\bar{x}, t_p + (x_p - \bar{x})/(\beta - 1)]\, d\bar{x}. \tag{19b}$$

† See, for example, Landau & Lifschitz (1960).

‡ $Y_w(x, t)$ is the value of Y at the wave boundary.

Here equations (16), (12*a*, *c*), and (15 −) have been used and we have set

$$\theta(x,t) = (R^2 U_b)_x.$$

For points $x < a-1$ the upper limit in (19*b*) may be replaced by $a-1$, since $\theta = 0$ for these values of x.

Fix x and take an $x_p < 0$. Then $Y \to R(x_p)$ as t (and therefore t_p) tends to infinity, since θ becomes exponentially small. The signals move horizontally and the values of U and H carried by them decay exponentially to 0 and $1-1/\beta$, according to § 5 (i).

On the other hand, if $0 < x_p < x_t$ and x is in the same range, $\theta \to -\frac{1}{2}(\beta-1)(R^2)_x$ [according to § 5 (ii)] and $Y \to R(x)$. For $x < 0$, $Y \to R(0) = 1$. The approach is algebraic. Signals follow the surface to its maximum section and then move away horizontally. All values of U between 0 and 2 and of H between $1-1/\beta$ and $1+1/\beta$ are carried, so that a combined vortex-current sheet is formed on the part $0 < x < x_t$ of the body.

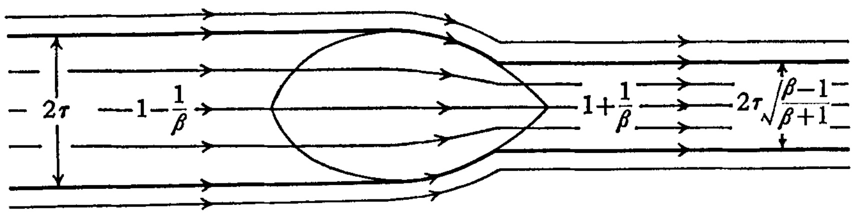

FIGURE 1. The ultimate magnetic field for $\beta = 2$ in case 1. The vertical scale has been greatly enlarged. Heavy lines are current sheets, outside which the flow field coincides with the magnetic field.

By changing the sign of β in equations (19) the ultimate form of the backward wave may be deduced. It stems from the part $x_t < x_p < a$ of the body and is attained exponentially. All signals move horizontally and carry the values $U = 0, H = 1+1/\beta$.

The ultimate potential disturbance now follows from equations (17) and (18). For $x < 0$ and $x > x_t$ they show that, on the wave boundary,

$$\phi_{xy} = 0.$$

In $0 < x < x_t$ the wave collapses and both x- and y-derivatives become infinite. However, omitting the t-derivative in equation (5 −) we may write

$$Y(1-\beta)U_x = -Y(V-\beta K)U_Y = -\theta U_Y$$

in equation (17*a*) and integrate formally. Equation (18*a*) then gives

$$\phi_{xy} = R',$$

on $y = \tau R(x)$, $0 < x < x_t$. The same result is obtained from equations (17*b*) and (18*b*).

The potential flow follows the two cylindrical wave regions and the connecting section of the body surface. The magnetic field parallels it. Figure 1 gives a sketch for $\beta = 2$.

All these results are similar to those for plane flow treated in L & L. Their present derivation is perhaps clearer. As there, the drag is all due to fluid pressure, and a similar calculation shows that the drag coefficient (based on the maximum cross-section) is $4(\beta-1)$.

8. Flow past a perfect conductor. Case 2

The front of the body ($x < 0$) always transmits upstream while the back ($x > 0$) always transmits downstream, since H_n does not vary in time. The boundary conditions (14) in conjunction with equations (15 ∓) show that

$$U = V = 0, \quad H = 1 \mp 1/\beta, \quad K = \mp (1/\beta) R' \quad (x \lessgtr 0)$$

on the body at all times. These values of U, V, H also hold in the wave regions, being achieved instantaneously across the wave fronts, while equation (17*b*) gives

$$K = \mp (1/\beta Y) RR'.$$

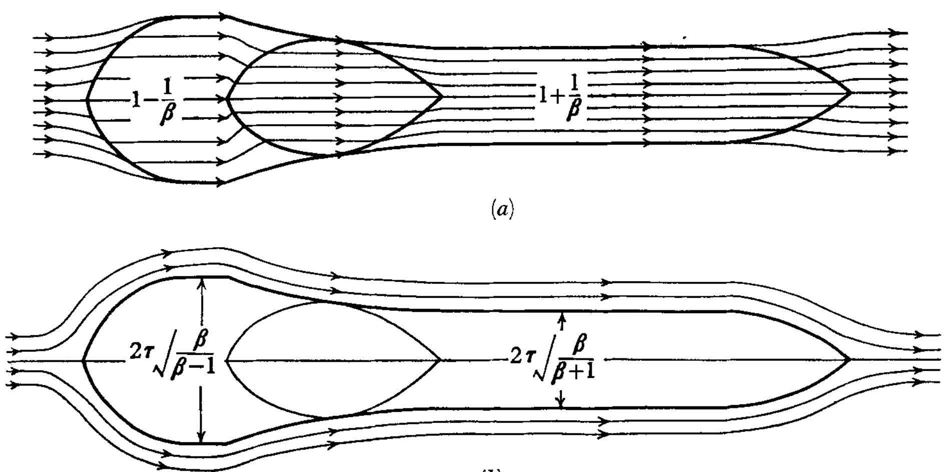

FIGURE 2. Wave regions for $\beta = 2$ when the body is perfectly conducting: (*a*) instantaneous magnetic field, heavy lines are current sheets, and also there are vertical current sheets in the wave regions at the leading and trailing edges; (*b*) instantaneous velocity field.

The trajectory of a signal in the forward wave can be given explicitly. It is determined by equations (19) with

$$\theta(x,t) = 2RR'$$

and therefore follows the curve

$$(\beta-1)\, Y^2 = \beta R^2(x_p) - R^2(x),$$

time being given by equation (19*a*). The wave front is ($t_p = 0$)

$$(\beta-1)\, Y^2 = \beta R^2[x+(\beta-1)t] - R^2(x), \tag{20a}$$

while the rest of the edge of the wave region is ($x_p = 0$)

$$(\beta-1)\, Y^2 = \beta - R^2(x). \tag{20b}$$

For the backward wave change the sign of β.

Upstream the flux tube through the body is spread from an initial radius τ to a final one $\surd\{\beta/(\beta-1)\}\tau$. Downstream it is reduced to a radius $\surd\{\beta/(\beta+1)\}\tau$. Once the wave fronts are completely developed they propagate unaltered. The only changes in the wave regions are then increases in the lengths of their central parts at the rates $(\beta \mp 1)\,t$ (see figure 2).

The compatibility of equations (18) is easily checked from equations (20). The potential flow instantaneously follows the shape formed by the two wave regions, as if they were solid. Ultimately it follows the two semi-infinite cylindrical parts of the wave regions and their join. The magnetic field then parallels it. Note that in addition to the edges of the wave regions there is a current sheet at the surface of the body.

The drag coefficient is 4β.

9. Flow past a perfect conductor. Case 3

The initial magnetic field is potential with vanishing normal component at the surface of the body. It is therefore parallel to the initial velocity field. This state is compatible with the boundary conditions so that no waves are emitted.

The initial fields persist for all time. H_n is always zero at the body, so that no signal can be emitted or received (or needs to be). Sears–Resler flow is achieved instantaneously.

Moreover, this result is not restricted to slender bodies (cf. Sears 1961) nor to values of β greater than 1.

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